Supervaluationism

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Readings

Suggested:

- Cobreros, Pablo & Tranchini, Luca (2019). *Supervaluationism, Subvaluationism and the Sorites Paradox*. In Sergi Oms & Elia Zardini (eds.), *The Sorites Paradox*. New York, NY: Cambridge University Press. pp. 38–62.

Outline

Making things precise

Bas van Fraassen

Kit Fine

- **Supervaluationism** (van Fraassen 1966; Fine 1975): handle vagueness by evaluating formulas over a range of *admissible precisifications*.
- A *precisification* is a classical "sharpening" of the vocabulary that preserves clear positives and clear negatives.
- Example: the predicate *heap* may be sharpened to "has at least n grains of sand," for many choices of n (e.g. $n = 1000, 1001, \ldots$).
- Thus, multiple precisifications are admissible: supervaluationism does not single out a *unique* cutoff.

Semantic Indecision

The reason it's vague where the outback begins is not that there's this thing, the outback, with imprecise borders; rather there are many things, with different borders, and nobody has been fool enough to try to enforce a choice of one of them as the official referent of the word "outback."
Vagueness is semantic indecision.

(Lewis 1986: *On the Plurality of Worlds*, p. 213)

[MacFarlane 2016]

Precisification

Definition (Precisification)

Let $v : P \to \{1, i, 0\}$ be a three-valued valuation. We say that a classical valuation v' is a *precisification* of v, and we write $v \leq v'$ iff:

$$v(p) = 1 \Rightarrow v'(p) = 1$$
$$v(p) = 0 \Rightarrow v'(p) = 0$$
$$v(p) = i \Rightarrow v'(p) \in \{0, 1\}$$

	p	q
v	i	0

	p	q
v'_1	1	0
v'_2	0	0

Supertrue and Superfalse

A formula is *supertrue* when it is true on *all* its precisifications; *superfalse* when it is false on all of them.

Definition (Supertruth & Superfalsity)

Let $v : P \to \{1, i, 0\}$ be three-valued and write $\mathrm{Prec}(v) := \{\, v' : v \leq v' \,\}$ for its set of classical precisifications. For any formula φ:

(Supertruth) $v \models^1 \varphi \iff \forall v' \in \mathrm{Prec}(v) : \; v'(\varphi) = 1$

(Superfalsity) $v \models^0 \varphi \iff \forall v' \in \mathrm{Prec}(v) : \; v'(\varphi) = 0$

Logical Consequence

We can define both a *global* and a *local* notion of consequence.

Definition (Global consequence)

$\Gamma \models_g \varphi$ iff for all three-valued valuations v, if $v \models^1 \gamma$ for all $\gamma \in \Gamma$, then $v \models^1 \varphi$.

Definition (Local consequence)

$\Gamma \models_l \varphi$ iff for all three-valued valuations v, for all $v' \in \mathrm{Prec}(v)$, if $v'(\gamma) = 1$ for all $\gamma \in \Gamma$, then $v'(\varphi) = 1$.

Local and Global

- Over the base propositional language $\{\neg, \wedge, \vee, \rightarrow\}$, global and local consequence coincide.
- In fact, supervaluationist consequence is equivalent to classical consequence.

Fact (Consequence equivalence)

$$\Gamma \models_g \varphi \text{ iff } \Gamma \models_l \varphi \text{ iff } \Gamma \models_{\mathrm{CL}} \varphi$$

Consequence equivalence (proof sketch)

$$\Gamma \models_g \varphi \;\Rightarrow\; \Gamma \models_l \varphi \;\Rightarrow\; \Gamma \models_{\mathrm{CL}} \varphi \;\Rightarrow\; \Gamma \models_g \varphi$$

For any classical $w : P \to \{0,1\}$, we can view w as three-valued (no i), so $\mathrm{Prec}(w) = \{w\}$. We write $v'(\Gamma) = 1$ for "$\forall \gamma \in \Gamma,\ v'(\gamma) = 1$".

(1) $\Gamma \models_g \varphi \Rightarrow \Gamma \models_l \varphi$**.** Let v be arbitrary and let $v' \in \mathrm{Prec}(v)$ with $v'(\Gamma) = 1$. Then $v' \models^1 \Gamma$ (since $\mathrm{Prec}(v') = \{v'\}$). By global consequence, $v' \models^1 \varphi$, hence $v'(\varphi) = 1$.

(2) $\Gamma \models_l \varphi \Rightarrow \Gamma \models_{\mathrm{CL}} \varphi$**.** Let w be classical with $w(\Gamma) = 1$. Viewing w as three-valued gives $\mathrm{Prec}(w) = \{w\}$. By local consequence, $w(\varphi) = 1$.

(3) $\Gamma \models_{\mathrm{CL}} \varphi \Rightarrow \Gamma \models_g \varphi$**.** Fix any three-valued v. If $v \models^1 \Gamma$, then for all $v' \in \mathrm{Prec}(v)$ we have $v'(\Gamma) = 1$. By classical consequence, $v'(\varphi) = 1$ for all such v', so $v \models^1 \varphi$.

Supervaluations as sets of valuations

- For a three-valued $v : P \to \{1, i, 0\}$, let $\mathrm{Prec}(v)$ be the set of all classical *precisifications* of v.
- Alternatively, start from an arbitrary nonempty set $V \subseteq \{0, 1\}^P$ of classical valuations and evaluate formulas pointwise over V.

Supertruth/superfalsity lift as:

$$V \models^1 \varphi \iff \forall v' \in V : \ v'(\varphi) = 1$$

$$V \models^0 \varphi \iff \forall v' \in V : \ v'(\varphi) = 0$$

This is equivalent to the original definition via v, taking $V = \mathrm{Prec}(v)$

Pointed evaluations

Given a non-empty set of classical valuations V[1], define the pointed satisfaction relation $V, v \models \varphi$ for $v \in V$ by:

$V, v \models p$	iff	$v(p) = 1$
$V, v \models \neg\varphi$	iff	$V, v \not\models \varphi$
$V, v \models \varphi \wedge \psi$	iff	$V, v \models \varphi$ and $V, v \models \psi$
$V, v \models \varphi \vee \psi$	iff	$V, v \models \varphi$ or $V, v \models \psi$
$V, v \models \varphi \to \psi$	iff	$V, v \not\models \varphi$ or $V, v \models \psi$

Definition (Supertruth)

Given a non-empty V, a formula φ is supertrue iff $V, v \models \varphi$ for all $v \in V$. We write $V \models^1 \varphi$.

Likewise, φ is superfalse iff $V \models^1 \neg\varphi$. We write $V \models^0 \varphi$.

[1] Allowing empty V does not change the resulting logic (as $\varnothing \models^1 \varphi$ for any φ), but it matters for satisfiability.

Local and Global

Likewise, global and local consequence can be recast in this way:

Definition (Global consequence)

$\Gamma \models_g \varphi$ iff for all non-empty V, if $V \models^1 \gamma$ for all $\gamma \in \Gamma$, then $V \models^1 \varphi$.

Definition (Local consequence)

$\Gamma \models_l \varphi$ iff for all non-empty V, for all $v \in V$, if $V, v \models \gamma$ for all $\gamma \in \Gamma$, then $V, v \models \varphi$.

Bivalence vs. Law of Excluded Middle

- **Failure of bivalence:** There are non-empty V and p with neither $V \models^1 p$ nor $V \models^1 \neg p$.
- **Validity of LEM:** Every classical tautology is supertrue. In particular for every non-empty V, $V \models^1 \; p \vee \neg p$.

Let $V = \{v_1, v_2\}$ with $v_1(p) = 1$ and $v_2(p) = 0$. Then

$$V \not\models^1 p \quad \text{and} \quad V \not\models^1 \neg p$$

since p fails at v_2 and $\neg p$ fails at v_1.

Modelling the Sorites

- Recall the descending Sorites sequence from p_1 to p_N.

$$\begin{array}{l} p_1 \\ p_1 \to p_2 \\ \vdots \\ p_{N-1} \to p_N \\ \hline p_N \end{array}$$

- A faithful model V for a descending Sorites series satisfies:
  - $V \models^1 p_1$
  - $V \models^0 p_N$
  - $\exists k \ (1 < k < N)$ with $V \not\models^1 p_k$ and $V \not\models^1 \neg p_k$
  - $\forall v \in V \ \forall m \in \{1, \ldots, N\}:$
    $$\begin{cases} v(p_m) = 1 \Rightarrow \forall k \ (1 \leq k \leq m \Rightarrow v(p_k) = 1) \\ v(p_m) = 0 \Rightarrow \forall k \ (m \leq k \leq N \Rightarrow v(p_k) = 0) \end{cases}$$

An example

- Take $N = 5$. A faithful model is:

V	p_1	p_2	p_3	p_4	p_5
v_1	1	0	0	0	0
v_2	1	1	0	0	0
v_3	1	1	1	0	0
v_4	1	1	1	1	0

$V \models^1 p_1$

$V \not\models^1 (p_1 \to p_2)$ and $V \not\models^0 (p_1 \to p_2)$

$V \not\models^1 (p_2 \to p_3)$ and $V \not\models^0 (p_2 \to p_3)$

$V \not\models^1 (p_3 \to p_4)$ and $V \not\models^0 (p_3 \to p_4)$

$V \not\models^1 (p_4 \to p_5)$ and $V \not\models^0 (p_4 \to p_5)$

$V \models^0 p_5$

An example (cut-off as a disjunction)

V	p_1	p_2	p_3	p_4	p_5
v_1	1	0	0	0	0
v_2	1	1	0	0	0
v_3	1	1	1	0	0
v_4	1	1	1	1	0

- $A : (p_1 \to p_2) \wedge (p_2 \to p_3) \wedge (p_3 \to p_4) \wedge (p_4 \to p_5)$
- $\neg A : (p_1 \wedge \neg p_2) \vee (p_2 \wedge \neg p_3) \vee (p_3 \wedge \neg p_4) \vee (p_4 \wedge \neg p_5)$
- A is superfalse, and $\neg A$ is supertrue. However, none of the disjuncts is supertrue (no specific cut-off point).

Assessing the situation

Supervaluationist answer to the Sorites: not all conditional premises are supertrue (so the argument is blocked), *without committing to which step* (no conditional is superfalse).

- In first-order guise: $\forall n\,(\varphi(n) \rightarrow \varphi(n{+}1))$ is superfalse (read: series of conjunctions), but $\exists n\,(\varphi(n) \wedge \neg\varphi(n{+}1))$ is supertrue (read: series of disjunctions).
- Yet for each particular d, $\{\varphi(d) \wedge \neg\varphi(d{+}1)\}$ is not supertrue. (No singled-out cut-off.)

The notion of truth

- Supervaluationism lifts truth from a single valuation to a set of valuations. This echoes two frameworks:
  1. **Modal logic:** formulas are evaluated relative to possible worlds. A formula is true in a model if it is true in **all worlds of the model**.
  2. **Team semantics:** formulas are evaluated w.r.t. a *team* (set of valuations). A formula is true in a model if it is true in **all valuations of the team**.
- Over the base language (without modal operators), both are equivalent to classical logic.
- But lifting truth to sets of valuations yields loss of bivalence.
- Adding modal operators (next) or defining different connectives yields logic whose consequence is different from classical propositional logic.

Team Semantics

- In team semantics (Hodges 1997, Väänänen 2007), the satisfaction relation uses a *possibly empty* set V of valuations and is defined over sets (not pointed):

$V \models p$	iff	$\forall v \in V:\ v(p) = 1$
$V \models \neg\varphi$	iff	$\forall v \in V:\ \{v\} \not\models \varphi$
$V \models \varphi \wedge \psi$	iff	$V \models \varphi$ and $V \models \psi$
$V \models \varphi \vee \psi$	iff	$\exists V', V''\ (V' \cup V'' = V,\ V' \models \varphi,\ V'' \models \psi)$

- Connection to supervaluationism (for the base language):

$$V \models \varphi \text{ iff } \forall v \in V:\ \{v\} \models \varphi \ \text{ (i.e. } v(\varphi) = 1)$$

- Note that this definition of disjunction (over possibly empty teams) gives classical disjunction:

$$V \models p \vee q \text{ iff } \forall v \in V:\ v(p) = 1 \text{ or } v(q) = 1$$

Outline

1. Supervaluationism

2. Modalized Supervaluationism

3. Higher-order Vagueness

4. Truth-functionality

Definitely operator

- Add a determinacy operator Δ ("definitely"). Think of Δ as a necessity operator where precisifications act as worlds.
- Intuitively, Δp is true at a precisification v' of v iff p holds at *all* precisifications of v.
- For simplicity, first take the accessibility relation to be universal:

$$V, v \models \Delta\varphi \quad \text{iff} \quad \forall v' \in V : \; V, v' \models \varphi$$

Deduction Theorem

$$p \models_g \Delta p$$

Let V be any non-empty set of valuations and assume $V \models^1 p$. Then p holds at every $v \in V$, hence $V \models^1 \Delta p$.

$$\not\models_g \; p \to \Delta p$$

Take $V = \{v_1, v_2\}$ with $v_1(p) = 1$ and $v_2(p) = 0$. Then $V, v_1 \models p$ but $V, v_1 \not\models \Delta p$ (since not all $v' \in V$ satisfy p). Hence $V, v_1 \not\models p \to \Delta p$. Therefore $p \to \Delta p$ is not globally valid.

So the deduction theorem fails: $\Gamma, \varphi \models_g \psi \;\not\Rightarrow\; \Gamma \models_g \; \varphi \to \psi$

Global vs. Local with Δ

$$\varphi \models_g \Delta\varphi$$

$$\varphi \not\models_l \Delta\varphi$$

- *Global holds:* If $V \models^1 p$ then $v(p) = 1$ for all $v \in V$, so $V, v \models \Delta p$ for all v. Hence $V \models^1 \Delta p$.
- *Local fails:* Take $V = \{v_1, v_2\}$ with $v_1(p) = 1$ and $v_2(p) = 0$. Then $V, v_1 \models p$ but $V, v_1 \not\models \Delta p$.

Semantics

- Formulas are evaluated not just wrt a set of valuations but a pair $M = \langle V, R \rangle$ with $V \neq \varnothing$ and $R \subseteq V \times V$. Each $v \in V$ is a classical valuation $v : P \to \{0, 1\}$.

$$
\begin{array}{lll}
M, v \models p & \text{iff} & v(p) = 1 \\
M, v \models \neg\varphi & \text{iff} & M, v \not\models \varphi \\
M, v \models \varphi \wedge \psi & \text{iff} & M, v \models \varphi \text{ and } M, v \models \psi \\
M, v \models \varphi \vee \psi & \text{iff} & M, v \models \varphi \text{ or } M, v \models \psi \\
M, v \models \varphi \to \psi & \text{iff} & M, v \not\models \varphi \text{ or } M, v \models \psi \\
M, v \models \Delta\varphi & \text{iff} & \forall v' \in V \; (\, vRv' \Rightarrow M, v' \models \varphi \,)
\end{array}
$$

Supertruth: $M \models^1 \varphi :\Longleftrightarrow \forall v \in V \; (M, v \models \varphi)$

Global: $\Gamma \models_g \varphi \iff \forall M \; (M \models^1 \Gamma \Rightarrow M \models^1 \varphi)$

Local (modal logic): $\Gamma \models_l \varphi \iff \forall M \, \forall v \in V (M, v \models \Gamma \Rightarrow M, v \models \varphi)$

Outline

Higher-order Vagueness

- Define indeterminate: $\nabla\varphi := \neg\Delta\varphi \wedge \neg\Delta\neg\varphi$.
- With *universal* accessibility:

$$\nabla\,\Delta p \text{ is not satisfiable.}$$

Frame constraints $\leftrightarrow$ modal axioms (for Δ)

$\mathbf{T}$ (reflexive)	$\Delta\varphi \rightarrow \varphi$
$\mathbf{4}$ (transitive)	$\Delta\varphi \rightarrow \Delta\Delta\varphi$
$\mathbf{B}$ (symmetric)	$\varphi \rightarrow \Delta\neg\Delta\neg\varphi$
$\mathbf{5}$ (euclidean)	$\neg\Delta\varphi \rightarrow \Delta\neg\Delta\varphi$
$\mathbf{S5}$ / equivalence	T+4+5; T+4+B; T+B+5 $\Rightarrow \Delta\Delta\varphi \vee \Delta\neg\Delta\varphi$ is valid.

- **To allow higher-order vagueness:** we need to drop some axioms. Which ones to keep for a 'Definitely' operator?

R is reflexive and transitive (S4-frames)

In reflexive and transitive frames, higher-order vagueness for Δ is *possible*: $\nabla\Delta p$ is satisfiable.

Let $M = \langle V, R\rangle$ with $V = \{a, b, c\}$ and

$$R = \{(x, x) \mid x \in V\} \;\cup\; \{(a, b), (a, c)\}$$

(This R is transitive: from aRb and bRb we get aRb; similarly for c.)
Valuation: $p(b) = 1$, $p(c) = 0$ (value at a arbitrary).

- $R[b] = \{b\}$, so $M, b \models \Delta p$.
- $R[c] = \{c\}$, so $M, c \not\models \Delta p$.
- Therefore $M, a \models \nabla\Delta p$.

Outline

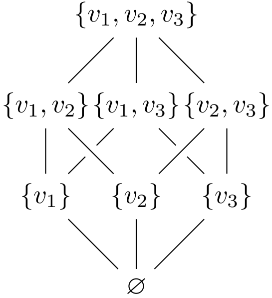
Supervaluations and truth-functionality

- **Truth-functionality:** the truth of a complex sentence is a function of the truth of its constituents.
- Supervaluationist theories are not truth-functional at the level of supertruth/superfalsity.
- For instance, the supertruth of $\neg p$ is not determined solely by whether p is supertrue.

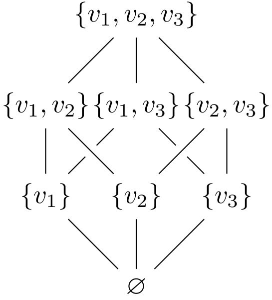
[Cobreros and Tranchini 2014; Field 2008]

An algebraic perspective

- Given V, the powerset $\mathcal{P}(V)$ identifies formulas with their set of supporting valuations.
- Take $V = \{v_1, v_2, v_3\}$ with $v_1(p) = 1$, $v_2(q) = 1$, others 0. Then p corresponds to $\{v_1\}$, and $p \vee q$ to $\{v_1, v_2\}$.



Functionality via sets

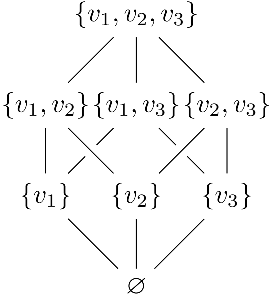


Functionality is preserved *extensionally*:

$$
\begin{array}{lcl}
f(p) & = & \{v \in V : \ v(p) = 1\} \\
f(\neg\varphi) & = & V \setminus f(\varphi) \\
f(\varphi \vee \psi) & = & f(\varphi) \cup f(\psi) \\
f(\varphi \wedge \psi) & = & f(\varphi) \cap f(\psi)
\end{array}
$$

φ is supertrue in V iff $f(\varphi) = V$; superfalse iff $f(\varphi) = \varnothing$.

Supervaluations and degrees



- This representation suggests a degree-theoretic flavor.
- However, take $V = \{v_1, v_2, v_3\}$ with $v_1(p) = 1$, $v_2(q) = 1$, others 0.
- Then $f(p) = \{v_1\}$ and $f(\neg p) = \{v_2, v_3\}$. But supervaluationism does *not* rank p as 'less true' than $\neg p$.

Discussion

- **Disjunction** can be supertrue without any supertrue disjunct, undermining the intuitive idea that a true disjunction is always made true by one of its disjuncts.
- While theoremhood over the base language is classical, familiar *argument forms* needn't be globally supervalid with Δ (e.g., conditional proof)
- Supertruth is not disquotational. If we identify ordinary truth with supertruth, the **T-schema is no longer valid** (though this might be an advantage for some).
- Higher-order vagueness is problematic with S5 axioms (plus a concern you have to address in your second assignment)
- Supervaluationists say "there is an n where the cutoff occurs" (since $\exists n\,(\varphi(n) \wedge \neg\varphi(n{+}1))$ is supertrue) yet deny there *really is* a sharp cutoff. Some see this as talking *as if* sharp boundaries exist.

Exercises

1. Show that (slide 25):

   (a) $\Gamma \models_l \varphi \quad \Rightarrow \quad \Gamma \models_g \varphi$

   (b) $\Gamma, \varphi \models_l \psi \quad \Leftrightarrow \quad \Gamma \models_l \varphi \rightarrow \psi$

   (c) $\Gamma \models_g \varphi \rightarrow \psi \quad \Rightarrow \quad \Gamma, \varphi \models_g \psi$

   (d) $\models_g \varphi \quad \Leftrightarrow \quad \models_l \varphi$

   (e) $\varphi \models_g \psi \quad \not\Rightarrow \quad \neg\psi \models_g \neg\varphi$

   (f) If R is not reflexive, then $\not\models (\Delta\varphi \rightarrow \varphi)$

   (g) $\nabla\Delta p$ is satisfiable on reflexive and symmetric frames

2. On the set-theoretic preservation of functionality (slide 32), add the clauses for:

   2.1 Universal Δ

   2.2 General Δ